

Mehler-Heine type formulas for Charlier and Meixner polynomials.

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Abstract

We derive Mehler–Heine type asymptotic formulas for Charlier and Meixner polynomials, and also for their associated families. These formulas provide good approximations for the polynomials in the neighborhood of $x = 0$, and determine the asymptotic limit of their zeros as the degree n goes to infinity.

Keywords: Mehler-Heine formulas, discrete orthogonal polynomials, associated polynomials, Stieltjes transforms

MSC-class: 41A30 (Primary), 33A65, 33A15, 44A15 (Secondary)

1 Introduction

Mehler–Heine type asymptotic formulas were introduced by Heinrich Eduard Heine (1821-1881) [31] and Gustav Ferdinand Mehler (1835-1895) [43]

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(see Watson's book [67, 5.71] for some historical remarks). They describe the asymptotic behavior of a family of orthogonal polynomials $P_n(x)$ as the degree n tends to infinity, near one edge of the support of the measure. They have the general form

$$\lim_{n \rightarrow \infty} f_n(x) P_n(A + u_n x) = g(x), \quad x \in S,$$

where S is a domain in the complex plane, A is a constant, u_n is a given sequence, $g(x)$ is analytic in S and the functions $f_n(x)$ are analytic and don't have any zeros in S , for sufficiently large n . The convergence is uniform on compact subsets of S . From Hurwitz's theorem [32, 4.10e], we conclude that for a fixed k

$$x_{n,k} \sim \frac{\zeta_k - A}{u_n}, \quad n \rightarrow \infty \quad (1)$$

where

$$x_{n,1} < x_{n,2} < \cdots < x_{n,n}$$

are the zeros of $P_n(x)$ and $\zeta_1 < \zeta_2 < \cdots$ are the zeros of $g(x)$.

Examples of Mehler-Heine formulas include the *Jacobi polynomials* $P_n^{(\alpha, \beta)}(x)$, defined by

$$P_n^{(\alpha, \beta)}(x) = \frac{(\alpha + 1)_n}{n!} {}_2F_1 \left(\begin{matrix} -n, n + \alpha + \beta + 1 \\ \alpha + 1 \end{matrix} ; \frac{1-x}{2} \right), \quad (2)$$

where $\alpha, \beta > -1$,

$${}_pF_q \left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} ; z \right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^k}{k!}$$

denotes the *Generalized Hypergeometric Function* and $(u)_k$ is the *Pochhammer symbol* (or rising factorial) [3],

$$(u)_k = u(u+1) \cdots (u+k-1).$$

For the Jacobi polynomials, we have [56]

$$\lim_{n \rightarrow \infty} n^{-\alpha} P_n^{(\alpha, \beta)} \left(1 - \frac{x^2}{2n^2} \right) = \left(\frac{x}{2} \right)^{-\alpha} J_{\alpha}(x), \quad (3)$$

where $J_{\alpha}(x)$ is the *Bessel function* of the first kind [3]

$$J_{\nu}(x) = \left(\frac{x}{2} \right)^{\nu} \frac{1}{\Gamma(\nu+1)} {}_0F_1 \left(\begin{matrix} - \\ \nu+1 \end{matrix} ; -\frac{x^2}{4} \right),$$

and $\Gamma(z)$ is the *Gamma function*. The case $\alpha = \beta = 0$ (Legendre polynomials) was the one originally considered by Mehler and Heine. Extensions of this result for some types of generalized Jacobi polynomials were studied in [27].

The *Laguerre polynomials* $L_n^{(\alpha)}(x)$, defined by [37]

$$L_n^{(\alpha)}(x) = \frac{(\alpha+1)_n}{n!} {}_1F_1\left(\begin{matrix} -n \\ \alpha+1 \end{matrix}; x\right), \quad \alpha > -1,$$

satisfy [56]

$$\lim_{n \rightarrow \infty} n^{-\alpha} L_n^{(\alpha)}\left(\frac{x^2}{4n}\right) = \left(\frac{x}{2}\right)^{-\alpha} J_{\alpha}(x).$$

Mehler-Heine type formulas for some classes of *multiple* (also called *poly-orthogonal*) [64] Jacobi and Laguerre polynomials were considered in [57], [20], and [54].

For the *Hermite polynomials* $H_n(x)$, defined by [37]

$$H_n(x) = (2x)^n {}_2F_0\left(\begin{matrix} -\frac{n}{2}, -\frac{n-1}{2} \\ - \end{matrix}; -\frac{1}{x^2}\right),$$

we have two distinct cases:

$$\lim_{n \rightarrow \infty} \frac{(-1)^n}{4^n n!} \sqrt{n} H_{2n}\left(\frac{x}{2\sqrt{n}}\right) = \left(\frac{x}{2}\right)^{\frac{1}{2}} J_{-\frac{1}{2}}(x),$$

and

$$\lim_{n \rightarrow \infty} \frac{(-1)^n}{4^n n!} H_{2n+1}\left(\frac{x}{2\sqrt{n}}\right) = (2x)^{\frac{1}{2}} J_{\frac{1}{2}}(x).$$

The Laguerre polynomials and the Hermite polynomials are related by the quadratic transformations [37]

$$\begin{aligned} H_{2n}(x) &= (-1)^n n! 2^{2n} L_n^{(-\frac{1}{2})}(x^2) \\ H_{2n+1}(x) &= (-1)^n n! 2^{2n+1} x L_n^{(\frac{1}{2})}(x^2). \end{aligned}$$

All of the Mehler-Heine formulas above can be derived from the result [37]

$$\lim_{\lambda \rightarrow \infty} {}_pF_q\left(\begin{matrix} a_1, \dots, a_{p-1}, \lambda a_p \\ b_1, \dots, b_q \end{matrix}; \frac{x}{\lambda}\right) = {}_{p-1}F_q\left(\begin{matrix} a_1, \dots, a_{p-1} \\ b_1, \dots, b_q \end{matrix}; a_p x\right).$$

In [4], A. Aptekarev generalized (3) in the following way:

Theorem 1 *Let $q_n(x)$ be an orthonormal system of polynomials defined by*

$$xq_n = b_n q_{n+1} + a_n q_n + b_{n-1} q_{n-1},$$

with

$$a_n \rightarrow 0, \quad b_n \rightarrow \frac{1}{2}, \quad (4)$$

and suppose that

$$\frac{q_{n+1}(1)}{q_n(1)} = 1 + \frac{\alpha + \frac{1}{2}}{n} + O(n^{-1}), \quad \alpha > -1.$$

Then,

$$\lim_{n \rightarrow \infty} \frac{1}{n^{\alpha + \frac{1}{2}}} q_n \left(1 - \frac{x^2}{2n^2} \right) = x^{-\alpha} J_{\alpha}(x),$$

uniformly on compact subsets of the complex plane.

The condition (4), indicates that the polynomials $q_n(x)$ are orthogonal with respect to a measure $\mu(x)$ that is supported on the interval $[-1, 1]$ and belongs to the *Nevai class* \mathcal{M} [46]. Aptekarev's result was extended in [60]. Similar results for multiple orthogonal polynomials were obtained in [58] and [61]

A somehow different type of example is provided by the *Modified Lommel Polynomials*, defined by [21]

$$h_{n,\nu}(x) = (\nu)_n (2x)^n {}_2F_3 \left(\begin{matrix} -\frac{n}{2}, \frac{1-n}{2} \\ \nu, -n, 1 - \nu - n \end{matrix} ; -\frac{1}{x^2} \right), \quad \nu > 0.$$

In this case,

$$\lim_{n \rightarrow \infty} \frac{(2x)^{1-\nu-n}}{\Gamma(n+\nu)} h_{n,\nu}(x) = J_{\nu-1}(x^{-1}), \quad x \neq 0,$$

uniformly on compact subsets of $\mathbb{C} \setminus \{0\}$. If we fix the value of x (say $x = \frac{1}{2}$), we obtain a different family of orthogonal polynomials in the variable ν [42]

$$R_n(z) = h_{n,z} \left(\frac{1}{2} \right).$$

For these polynomials, we have

$$\lim_{n \rightarrow \infty} \frac{1}{\Gamma(n+z)} R_n(z) = J_{z-1}(2).$$

Generalizations of Mehler-Heine type formulas in the context of Riemannian geometry were given in [7], [8], [18], and [55]. Extensions to polynomials in several variables were studied in [12].

Mehler-Heine type formulas have been extensively used in the theory of *Sobolev orthogonal polynomials* see (among many other articles) [2], [13], [15], [19], [25], [28], and [44].

The work presented in this paper was motivated by a question that Professor Juan José Moreno-Balcázar asked during our visit to the Universidad de Almería in 2010. He was wondering if it would be possible to have Mehler-Heine type formulas for discrete orthogonal polynomials, since this would help in calculations involving asymptotics of discrete Sobolev polynomials [45].

The answer is affirmative, and we have obtained results for the Charlier and Meixner polynomials. These are the only two infinite families of classical orthogonal polynomials in the discrete lattice $\{0, 1, 2, \dots\}$.

2 Preliminaries

Let $\psi(t)$ be a bounded, non-decreasing function on \mathbb{R} , with finite *moments*

$$\mu_n = \int_{\mathbb{R}} t^n d\psi(t) < \infty, \quad n = 0, 1, \dots \quad (5)$$

and assume that the set

$$\mathfrak{S}(\psi) = \{t \in \mathbb{R} \mid \psi(t + \delta) - \psi(t - \delta) > 0 \quad \forall \delta > 0\}$$

(called the *spectrum* of ψ) is infinite. Under these assumptions, there exists a unique sequence of monic polynomials $\widehat{P}_n(x)$, with $\deg \widehat{P}_n = n$, such that

$$\int_{\mathbb{R}} \widehat{P}_n(t) \widehat{P}_m(t) d\psi(t) = K_n \delta_{n,m}, \quad K_n > 0.$$

The polynomials $\widehat{P}_n(x)$ satisfy a *three-term recurrence relation*

$$x \widehat{P}_n = \widehat{P}_{n+1} + b_n \widehat{P}_n + c_n \widehat{P}_{n-1}, \quad (6)$$

with initial conditions $\widehat{P}_{-1}(x) = 0$, $\widehat{P}_0(x) = 1$.

The inverse problem of finding a distribution function $\psi(t)$ satisfying (5) is called the *Hamburger moment problem* [1], [52], [53]. The moment problem is called *determinate* if there exists a unique solution, and *indeterminate* otherwise [62], [10]. A possible criterion for the determinacy of the problem is due to Carleman [17]. Carleman's Theorem says that the problem is determinate if

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{c_n}} = \infty, \quad (7)$$

where the coefficients $c_n > 0$ were defined in (6).

The *associated orthogonal polynomials* $P_n^*(x)$ are defined by [50]

$$xP_n^* = P_{n+1}^* + b_n P_n^* + c_n P_{n-1}^*, \quad P_0^* = 0, \quad P_1^* = 1.$$

Note that $\deg P_n^*(x) = n - 1$. We have [16]

$$\mu_0 P_n^*(x) = \int_{\mathbb{R}} \frac{\widehat{P}_n(x) - \widehat{P}_n(t)}{x - t} d\psi(t),$$

where μ_0 was defined in (5). The associated classical discrete orthogonal polynomials were studied in [6], [22], [29], [34], [39], [40], and [65].

The connection between $\widehat{P}_n(x)$, $P_n^*(x)$ and the distribution function $\psi(t)$ is given by *Markov's theorem*:

$$\lim_{n \rightarrow \infty} \mu_0 \frac{P_n^*(z)}{\widehat{P}_n(z)} = \int_{\mathbb{R}} \frac{d\psi(t)}{z - t}, \quad z \notin \Lambda, \quad (8)$$

where $\Lambda = [\inf(\mathfrak{S}), \sup(\mathfrak{S})]$, and the convergence is uniform in compact subsets of $\mathbb{C} \setminus \Lambda$. The original theorem was proved when Λ is a finite interval, but it is also true as long as the corresponding Hamburger moment problem is determinate [9], [63].

The function

$$S(z) = \int_{\mathbb{R}} \frac{d\psi(t)}{z - t}, \quad z \notin \Lambda,$$

is called the *Stieltjes transform* of $\psi(t)$. For the class of discrete distributions that we are considering, we have the following result.

Lemma 2 *The Stieltjes transform of the distribution*

$$\psi(t) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{c^k}{k!} u(t-k),$$

where $u(t)$ is the unit step function

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases},$$

is given by

$$S(z) = \frac{1}{z} {}_{p+1}F_{q+1} \left(\begin{matrix} -z, a_1, \dots, a_p \\ 1-z, b_1, \dots, b_q \end{matrix}; c \right), \quad z \in \mathbb{C} \setminus [0, \infty). \quad (9)$$

Proof. Since for all $z \in \mathbb{C} \setminus [0, \infty)$

$$\frac{(-z)_k}{(1-z)_k} = \prod_{j=0}^{k-1} \frac{-z+j}{1-z+j} = \frac{z}{z-k},$$

we have

$$S(z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{c^k}{k!} \frac{1}{z-k} = \frac{1}{z} \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{c^k}{k!} \frac{(-z)_k}{(1-z)_k},$$

and the result follows. ■

Remark 3 *From the representation*

$$S(z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{c^k}{k!} \frac{1}{z-k},$$

it is clear that $S(z)$ is a meromorphic function with simple poles at $z = 0, 1, \dots$, and simple zeros between them. Since the reciprocal Gamma function is an entire function with simple zeros at $z = 0, -1, \dots$, we see that $\frac{S(z)}{\Gamma(-z)}$ is an entire function with infinitely many simple zeros located in the intervals $(n, n+1)$, $n = 0, 1, \dots$.

The following result is known as *Tannery's theorem* [59]. Although there are many proofs available in the literature [14], [33], [36], we include one for the sake of completeness.

Theorem 4 *Suppose that we have*

$$l_k \leq a_k(n) \leq u_k, \quad 0 \leq k \leq n,$$

that

$$\lim_{n \rightarrow \infty} a_k(n) = A_k, \quad k = 0, 1, \dots,$$

and that

$$\sum_{k=0}^{\infty} l_k, \quad \sum_{k=0}^{\infty} A_k, \quad \sum_{k=0}^{\infty} u_k$$

are all convergent series. Then,

$$\lim_{n \rightarrow \infty} \sum_{k=0}^n a_k(n) = \sum_{k=0}^{\infty} A_k.$$

Proof. Let $p < n$ be natural numbers and

$$x_n = \sum_{k=0}^n a_k(n).$$

Then,

$$\sum_{k=0}^p a_k(n) + \sum_{k=p+1}^n l_k \leq x_n \leq \sum_{k=0}^p a_k(n) + \sum_{k=p+1}^n u_k.$$

Letting $n \rightarrow \infty$, we get

$$\sum_{k=0}^p A_k + \sum_{k=p+1}^{\infty} l_k \leq \varliminf_{n \rightarrow \infty} x_n \leq \overline{\lim}_{n \rightarrow \infty} x_n \leq \sum_{k=0}^p A_k + \sum_{k=p+1}^{\infty} u_k.$$

But since $\sum_{k=0}^{\infty} l_k, \sum_{k=0}^{\infty} A_k, \sum_{k=0}^{\infty} u_k$ converge, we can let $p \rightarrow \infty$, and obtain

$$\sum_{k=0}^{\infty} A_k \leq \varliminf_{n \rightarrow \infty} x_n \leq \overline{\lim}_{n \rightarrow \infty} x_n \leq \sum_{k=0}^{\infty} A_k.$$

Thus,

$$\lim_{n \rightarrow \infty} x_n = \sum_{k=0}^{\infty} A_k.$$

■

3 Charlier

The *Charlier polynomials* $C_n(x; a)$ are defined by [37]

$$C_n(x; a) = {}_2F_0 \left(\begin{matrix} -n, -x \\ - \end{matrix} ; -\frac{1}{a} \right), \quad (10)$$

with $a > 0$, and the corresponding monic polynomials are

$$\widehat{C}_n(x; a) = (-a)^n C_n(x; a). \quad (11)$$

The Charlier polynomials are orthogonal with respect to the distribution

$$\psi(t) = \sum_{k=0}^{\infty} \frac{a^k}{k!} u(t - k),$$

and satisfy

$$\sum_{k=0}^{\infty} C_n(k; a) C_m(k; a) \frac{a^k}{k!} = a^{-n} e^a n! \delta_{n,m},$$

where $\delta_{n,m}$ is Kronecker's delta

$$\delta_{n,m} = \begin{cases} 1, & n = m \\ 0, & n \neq m \end{cases}.$$

Note that if we set $n = m = 0$ we get

$$\mu_0 = \sum_{k=0}^{\infty} \frac{a^k}{k!} = e^a. \quad (12)$$

The Mehler-Heine type formula for the Charlier polynomials is the following.

Proposition 5 *For all complex numbers x , we have*

$$\lim_{n \rightarrow \infty} \frac{a^n}{\Gamma(n - x)} C_n(x; a) = \frac{e^a}{\Gamma(-x)}. \quad (13)$$

Proof. From the hypergeometric representation (10), we get

$$\frac{a^n}{(-x)_n} C_n(x; a) = a^n \sum_{j=0}^n \frac{(-n)_j (-x)_j}{(1)_j (-x)_n} (-a)^{-j}.$$

Changing the summation variable to $k = n - j$, we have

$$\frac{a^n}{(-x)_n} C_n(x; a) = a^n \sum_{k=0}^n \frac{(-n)_{n-k} (-x)_{n-k}}{(1)_{n-k} (-x)_n} (-a)^{k-n}. \quad (14)$$

Using the identity [47, 18:5:10]

$$(s)_l = (s)_m (s+m)_{l-m}, \quad m = 0, 1, \dots, \quad (15)$$

with $s = -x$, $l = n$ and $m = n - k$, we get

$$\frac{(-x)_{n-k}}{(-x)_n} = \frac{1}{(n-k-x)_k}.$$

From the formula [47, 18:5:1]

$$(-s)_l = (-1)^l (s-l+1)_l, \quad (16)$$

with $s = n$ and $l = n - k$, we have

$$(-n)_{n-k} = (-1)^{n-k} (k+1)_{n-k}.$$

Thus, we can rewrite (14) as

$$\frac{a^n}{(-x)_n} C_n(x; a) = \sum_{k=0}^n \frac{(k+1)_{n-k}}{(1)_{n-k}} \frac{a^k}{(n-k-x)_k}. \quad (17)$$

Using the identity [47, 18:5:1]

$$\frac{(s+m)_l}{(s)_l} = \frac{(s+l)_m}{(s)_m}, \quad m = 0, 1, \dots \quad (18)$$

with $s = 1$, $l = n - k$ and $m = k$ in (17), we obtain

$$\frac{a^n}{(-x)_n} C_n(x; a) = \sum_{k=0}^n \frac{(n-k+1)_k}{(1)_k} \frac{a^k}{(n-k-x)_k}. \quad (19)$$

But clearly, for all $0 \leq k \leq n$, with $x \leq -1$,

$$0 < \frac{(n-k+1)_k}{(n-k-x)_k} = \prod_{j=0}^{k-1} \frac{n-k+1+j}{n-k-x+j} \leq 1,$$

and for all $k = 0, 1, \dots$

$$\lim_{n \rightarrow \infty} \frac{(n - k + 1)_k}{(n - k - x)_k} = 1, \quad x \leq -1.$$

Therefore, from Tannery's theorem we conclude that

$$\lim_{n \rightarrow \infty} \frac{a^n}{(-x)_n} C_n(x; a) = \sum_{k=0}^{\infty} \frac{a^k}{k!} = e^a, \quad x \leq -1. \quad (20)$$

Dividing both sides of (20) by $\Gamma(-x)$, we have

$$\lim_{n \rightarrow \infty} \frac{a^n}{\Gamma(n - x)} C_n(x; a) = \frac{e^a}{\Gamma(-x)}, \quad x \leq -1. \quad (21)$$

However, since both sides of the equation are analytic in the whole complex plane, it follows from the principle of analytic continuation that the formula is valid for all x . ■

Other types of asymptotic approximations for $C_n(x; a)$ as $n \rightarrow \infty$ were given in [11], [23], [26], [30], and [49].

3.1 Associated polynomials

The monic Charlier polynomials satisfy the three-term recurrence relation [37]

$$x\widehat{C}_n = \widehat{C}_{n+1} + (n + a)\widehat{C}_n + an\widehat{C}_{n-1},$$

with initial conditions

$$\widehat{C}_{-1}(x; a) = 0, \quad \widehat{C}_0(x; a) = 1.$$

The associated polynomials $C_n^*(x; a)$ satisfy the same recurrence, but the initial conditions are

$$C_0^*(x; a) = 0, \quad C_1^*(x; a) = 1.$$

Using Carleman's Theorem (7), we see that the moment problem is determinate. Hence, from (8) and (9) we have

$$\lim_{n \rightarrow \infty} e^a \frac{C_n^*(z; a)}{\widehat{C}_n(z; a)} = \frac{1}{z} {}_1F_1 \left(\begin{matrix} -z \\ 1 - z \end{matrix} ; a \right), \quad z \in \mathbb{C} \setminus [0, \infty) \quad (22)$$

where we have used (12).

From (11) and (13), it follows that

$$\lim_{n \rightarrow \infty} \frac{(-1)^n}{\Gamma(n-x)} \widehat{C}_n(x; a) = \frac{e^a}{\Gamma(-x)}. \quad (23)$$

Thus, from (22) and (23) we obtain

$$\lim_{n \rightarrow \infty} \frac{(-1)^n}{\Gamma(n-x)} C_n^*(x; a) = \frac{1}{x\Gamma(-x)} {}_1F_1 \left(\begin{matrix} -x \\ 1-x \end{matrix}; a \right). \quad (24)$$

Using the formulas [48, 13.6.5, 8.2.6]

$${}_1F_1 \left(\begin{matrix} b \\ b+1 \end{matrix}; -z \right) = bz^{-b} \gamma(b, z) = b\Gamma(b) \gamma^*(b, z),$$

we get

$$\lim_{n \rightarrow \infty} \frac{(-1)^n}{\Gamma(n-x)} C_n^*(x; a) = -\gamma^*(-x, -a), \quad (25)$$

where $\gamma^*(b, z)$ is the *entire incomplete gamma function* defined by [48, 8.2.7]

$$\gamma^*(b, z) = \frac{1}{\Gamma(b)} \int_0^1 t^{b-1} e^{-zt} dt,$$

for $\text{Re}(b) > 0$, and by analytic continuation elsewhere. The function $\gamma^*(b, z)$ is entire in b and z , and has two zeros in each of the intervals $(2n-2, 2n)$ for all $n = 1, 2, \dots$ [38]. It follows from (1) that the zeros of $C_n^*(x; a)$ approach the zeros of the function $\gamma^*(-x, -a)$ as $n \rightarrow \infty$.

Using the formula [47, 45:6:4]

$$\gamma^*(b, z) = e^{-z} \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(b+1+j)},$$

we can rewrite (24) as

$$\lim_{n \rightarrow \infty} \frac{(-1)^n}{\Gamma(n-x)} C_n^*(x; a) = -e^a \sum_{k=0}^{\infty} \frac{(-a)^k}{\Gamma(1-x+k)}.$$

In Figure 1, we plot the functions

$$\frac{1}{\Gamma(28-x)} C_{28}^*(x; 1.23), \quad -\gamma^*(-x, -1.23),$$

to illustrate the accuracy of (25).

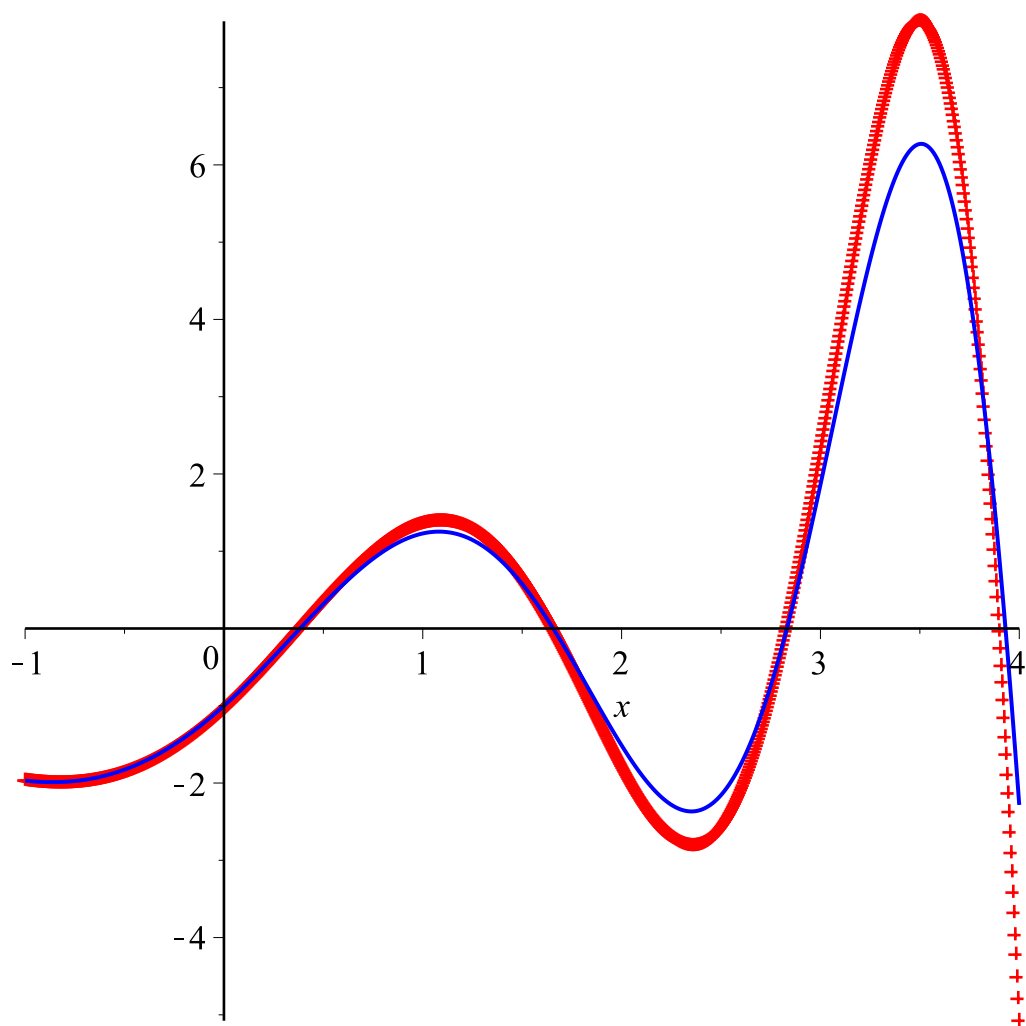


Figure 1: A plot of the scaled polynomial C_{28}^* (+++) and the limiting function (solid line).

3.2 Meixner

The *Meixner polynomials* $M_n(x; \beta, c)$ are defined by [37]

$$M_n(x; \beta, c) = {}_2F_1 \left(\begin{matrix} -n, -x \\ \beta \end{matrix}; 1 - \frac{1}{c} \right), \quad (26)$$

where $\beta > 0$ and $0 < c < 1$. The monic polynomials are

$$\widehat{M}_n(x; \beta, c) = (\beta)_n \left(\frac{c}{c-1} \right)^n M_n(x; \beta, c). \quad (27)$$

The Meixner polynomials are orthogonal with respect to the distribution

$$\psi(t) = \sum_{k=0}^{\infty} (\beta)_k \frac{c^k}{k!} u(t-k),$$

and satisfy [37]

$$\sum_{k=0}^{\infty} M_n(x; \beta, c) M_m(x; \beta, c) (\beta)_k \frac{c^k}{k!} = \frac{c^{-n} n!}{(\beta)_n (1-c)^\beta} \delta_{n,m}.$$

In particular, for $n = m = 0$, we get

$$\mu_0 = \sum_{k=0}^{\infty} (\beta)_k \frac{c^k}{k!} = (1-c)^{-\beta}. \quad (28)$$

The Mehler-Heine type formula for the Meixner polynomials is the following.

Proposition 6 *For all complex numbers x , we have*

$$\lim_{n \rightarrow \infty} \frac{c^n (\beta)_n}{\Gamma(n-x)} M_n(x; \beta, c) = \frac{1}{(1-c)^{\beta+x} \Gamma(-x)}. \quad (29)$$

Proof. From (26) and the formula [3, 2.3.14]

$${}_2F_1 \left(\begin{matrix} -n, a \\ b \end{matrix}; x \right) = \frac{(b-a)_n}{(b)_n} {}_2F_1 \left(\begin{matrix} -n, a \\ a+1-n-b \end{matrix}; 1-x \right),$$

we get

$$M_n(x; \beta, c) = \frac{(x + \beta)_n}{(\beta)_n} {}_2F_1 \left(\begin{matrix} -n, -x \\ -x + 1 - n - \beta \end{matrix}; \frac{1}{c} \right).$$

Thus,

$$(\beta)_n c^n M_n(x; \beta, c) = (x + \beta)_n c^n {}_2F_1 \left(\begin{matrix} -n, -x \\ -x + 1 - n - \beta \end{matrix}; \frac{1}{c} \right).$$

It follows that

$$(\beta)_n (-x)_n c^n M_n(x; \beta, c) = \sum_{k=0}^n \frac{(-n)_k (-x)_k (x + \beta)_n (-x)_n c^{n-k}}{(-x + 1 - n - \beta)_k k!},$$

or

$$(\beta)_n c^n M_n(x; \beta, c) = \sum_{k=0}^n \frac{(-n)_{n-k} (-x)_{n-k} (x + \beta)_n}{(-x + 1 - n - \beta)_{n-k}} \frac{c^k}{(n-k)!}. \quad (30)$$

Using (16) and (15) we have

$$\frac{(-n)_{n-k}}{(-x + 1 - n - \beta)_{n-k}} = \frac{(k+1)_{n-k}}{(x + \beta + k)_{n-k}} = \frac{(1)_n (x + \beta)_k}{(1)_k (x + \beta)_n}.$$

Hence, we can write (30) in the form

$$(\beta)_n c^n M_n(x; \beta, c) = \sum_{k=0}^n \frac{(1)_n}{(1)_k} (x + \beta)_k (-x)_{n-k} \frac{c^k}{(n-k)!}, \quad (31)$$

and therefore

$$\frac{(\beta)_n c^n M_n(x; \beta, c)}{(-x)_n} = \sum_{k=0}^n \frac{(-x)_{n-k}}{(-x)_n} \frac{n!}{(n-k)!} (x + \beta)_k \frac{c^k}{k!}. \quad (32)$$

But since

$$\frac{(-x)_{n-k}}{(-x)_n} \frac{n!}{(n-k)!} = \prod_{j=0}^{k-1} \frac{n-j}{n-j-(x+1)} \leq 1, \quad x \leq -1,$$

and

$$\lim_{n \rightarrow \infty} \frac{(-x)_{n-k}}{(-x)_n} \frac{n!}{(n-k)!} = 1, \quad x \leq -1,$$

we can use Tannery's theorem and conclude that

$$\lim_{n \rightarrow \infty} \frac{(\beta)_n c^n M_n(x; \beta, c)}{(-x)_n} = \sum_{k=0}^{\infty} (x + \beta)_k \frac{c^k}{k!} = (1 - c)^{-x-\beta}, \quad x \leq -1. \quad (33)$$

Dividing both sides of (33) by $\Gamma(-x)$, we have

$$\lim_{n \rightarrow \infty} \frac{c^n (\beta)_n}{\Gamma(n - x)} M_n(x; \beta, c) = \frac{1}{(1 - c)^{\beta+x} \Gamma(-x)}, \quad x \leq -1.$$

Since both sides of the equation are analytic in the whole complex plane, it follows from the principle of analytic continuation that the formula is valid for all x . ■

Other asymptotic approximations for $M_n(x; \beta, c)$ as $n \rightarrow \infty$ were studied in [5], [35], [41], [51], [66], and [68].

3.3 Associated polynomials

The monic Meixner polynomials satisfy the three-term recurrence relation [37]

$$x \widehat{M}_n = \widehat{M}_{n+1} + \frac{n + (n + \beta)c}{1 - c} \widehat{M}_n + \frac{n(n + \beta - 1)c}{(1 - c)^2} \widehat{M}_{n-1},$$

with initial conditions

$$\widehat{M}_{-1}(x; \beta, c) = 0, \quad \widehat{M}_0(x; \beta, c) = 1.$$

The associated polynomials $M_n^*(x; \beta, c)$ satisfy the same recurrence, but the initial conditions are

$$M_0^*(x; \beta, c) = 0, \quad M_1^*(x; \beta, c) = 1.$$

Using Carleman's Theorem (7), we see that the moment problem is determinate. Hence, from (8) and (9) we have

$$\lim_{n \rightarrow \infty} (1 - c)^{-\beta} \frac{M_n^*(z; \beta, c)}{\widehat{M}_n(z; \beta, c)} = \frac{1}{z} {}_2F_1 \left(\begin{matrix} -z, \beta \\ 1 - z \end{matrix}; c \right), \quad z \in \mathbb{C} \setminus [0, \infty), \quad (34)$$

where we have used (28).

From (27) and (29), it follows that

$$\lim_{n \rightarrow \infty} \frac{(c-1)^n}{\Gamma(n-x)} \widehat{M}_n(x; \beta, c) = \frac{1}{(1-c)^{\beta+x} \Gamma(-x)}. \quad (35)$$

Thus, from (34) and (35) we obtain

$$\lim_{n \rightarrow \infty} \frac{(c-1)^n}{\Gamma(n-x)} M_n^*(x; \beta, c) = \frac{(1-c)^{-x}}{x \Gamma(-x)} {}_2F_1 \left(\begin{matrix} -x, \beta \\ 1-x \end{matrix}; c \right). \quad (36)$$

Using the formula [48, 8.17.7]

$${}_2F_1 \left(\begin{matrix} a, 1-b \\ a+1 \end{matrix}; z \right) = az^{-a} B_z(a, b),$$

we get

$$\lim_{n \rightarrow \infty} \frac{(c-1)^n}{\Gamma(n-x)} M_n^*(x; \beta, c) = - \left(\frac{c}{1-c} \right)^x \frac{1}{\Gamma(-x)} B_c(-x, 1-\beta),$$

where $B_z(a, b)$ is the *incomplete Beta function* defined by [47, 58:3:5]

$$B_z(a, b) = z^a \int_0^1 t^{a-1} (1-zt)^{b-1} dt,$$

for $a, b > 0$, $z \in [0, 1]$, and by analytic continuation elsewhere. It follows from (1) that the zeros of $M_n^*(x; \beta, c)$ approach the zeros of the function $B_c(-x, 1-\beta)$ as $n \rightarrow \infty$.

In Figure 2, we plot the functions

$$\frac{(c-1)^{28}}{\Gamma(28-x)} M_{28}^*(x; 1.23, 0.36), \quad \frac{(1-0.36)^{-x}}{x \Gamma(-x)} {}_2F_1 \left(\begin{matrix} -x, 1.23 \\ 1-x \end{matrix}; 0.36 \right),$$

to illustrate the accuracy of (36).

4 Conclusion

We have derived Mehler-Heine type formulas for the Charlier and Meixner families and their associated polynomials. We plan to extend this investigation to include other discrete orthogonal polynomials of class one [24].

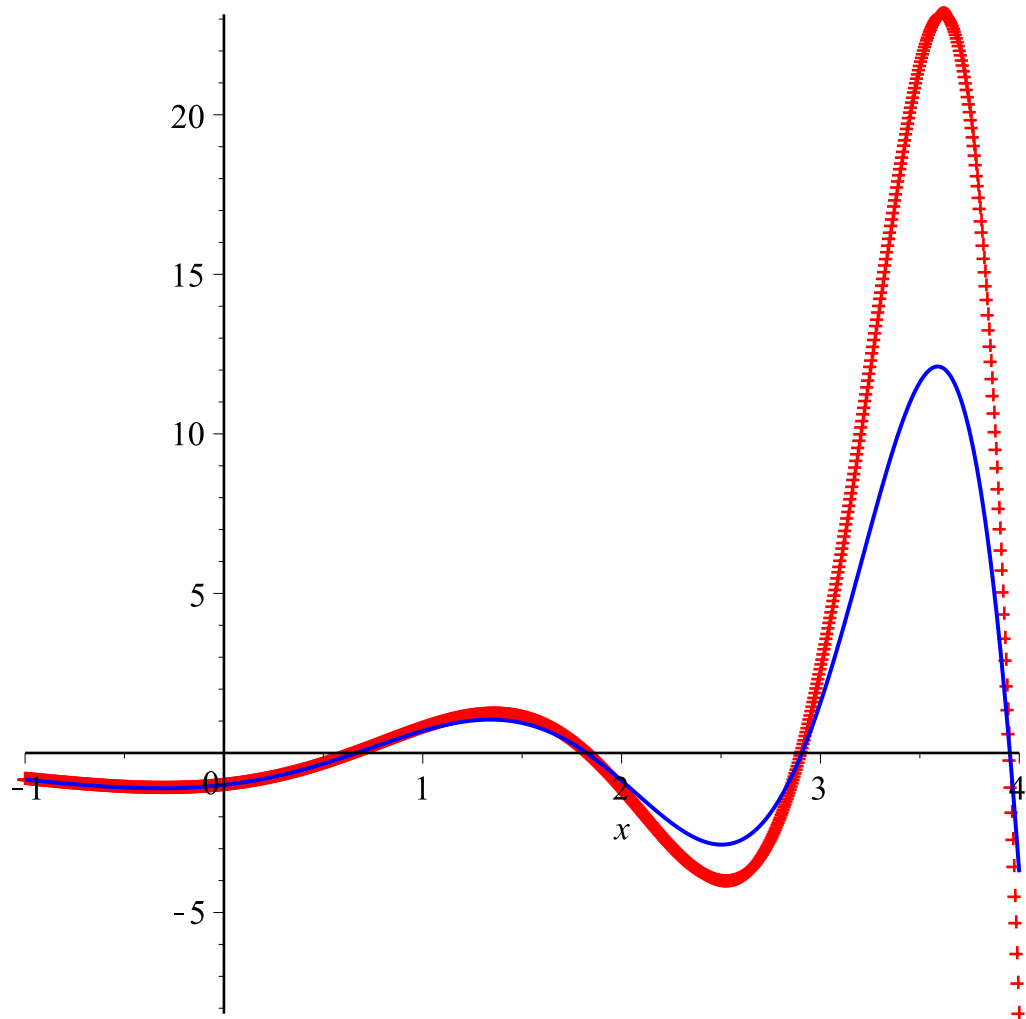


Figure 2: A plot of the scaled polynomial M_{28}^* (+++) and the limiting function (solid line).

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